

The Weighted Bergman Kernel and the Green's Function

by STEVEN KRANTZ (Washington Univ. at St. Louis)
and PAWEŁ M. WÓJCICKI (Warsaw)

Abstract

We study the connection between weighted Bergman kernel and Green's function on a domain $W \subset \mathbb{C}$ for which the Green's function exists.

1 Introduction

The Bergman kernel (see for instance [1, 9, 10, 16, 20]) has become a very important tool in geometric function theory, both in one and several complex variables. It turns out that not only the classical Bergman kernel, but also the weighted one can be useful (see [4, 5, 12] for instance). Let $W \subset \mathbb{C}$ be a domain, such that the Bergman space $L_H^2(W)$ is a non-zero space and G_W the Green's function of W (let us recall that G_W exists if $\mathbb{C} \setminus W$ is not polar, and this is only if $L_H^2(W) \neq 0$ —see [11] and [3]).

It is known, in the classical case, that

$$K_W(z, w) = -\frac{2}{\pi} \frac{\partial^2}{\partial z \partial \bar{w}} G_W(z, w)$$

(see [21]) for $z, w \in W$, $z \neq w$ (it was originally proved in [2] with additional assumptions on ∂W). On the other hand, if ∂W consists of a finite number of Jordan curves, $\rho(z)$ is a positive continuously differentiable function of x and y on a neighborhood of \bar{W} , $K_{W,\rho}(z, w)$ a weighted Bergman kernel of the space $L_H^2(W, \rho)$ and $G_{W,\rho}$ the Green's function for an operator $P_\rho = \frac{\partial}{\partial \bar{z}} \frac{1}{\rho(z)} \frac{\partial}{\partial z}$, then

$$K_{W,\rho}(z, w) = -\frac{2}{\pi \rho(z) \rho(w)} \frac{\partial^2}{\partial z \partial \bar{w}} G_{W,\rho}(z, w)$$

2010 *Mathematics Subject Classification*: Primary 32A36; Secondary 32A25.

Key words and phrases: weighted Bergman kernel; Green's function; harmonic function.

(see [6]). In this paper we prove that the connection above holds for any domain $W \subset \mathbb{C}$, for which $L_H^2(W) \neq 0$. Along the way, we give a shorter proof of the connection for unweighted kernels given in [21] in the case when W is bounded.

We shall begin with the definitions and basic facts used in this paper. Additionally, because we are dealing with the weighted Bergman kernels, we will recall for which weights in general the weighted Bergman kernel exists (although we are working here with differentiable weights only).

2 Definitions and Notation

Let $W \subset \mathbb{C}$ be a domain, and let $\mathcal{W}(W)$ be the set of weights on W , i.e., $\mathcal{W}(W)$ is the set of all Lebesgue measurable, real-valued, positive functions μ on W (we consider two weights as equivalent if they are equal almost everywhere with respect to the Lebesgue measure on W). For $\mu \in \mathcal{W}(W)$ we denote by $L^2(W, \mu)$ the space of all Lebesgue measurable, complex-valued, μ -square integrable functions on W , equipped with the norm $\|\cdot\|_{W, \mu} := \|\cdot\|_\mu$ given by the scalar product

$$\langle f|g \rangle_\mu := \int_W f(z) \overline{g(z)} \mu(z) dV, \quad f, g \in L^2(W, \mu).$$

The space $L_H^2(W, \mu) = \text{Hol}(W) \cap L^2(W, \mu)$ is called the *weighted Bergman space*, where $\text{Hol}(W)$ denotes the space of all holomorphic functions on the domain W . For any $z \in W$ we define the evaluation functional E_z on $L_H^2(W, \mu)$ by the formula

$$E_z f := f(z), \quad f \in L_H^2(W, \mu).$$

Let us recall the definition [Def. 2.1] of admissible weight given in [14].

Definition 2.1 (Admissible weight). A weight $\mu \in \mathcal{W}(W)$ is called an *admissible weight*, an *a-weight* for short, if $L_H^2(W, \mu)$ is a closed subspace of $L^2(W, \mu)$ and for any $z \in W$, the evaluation functional E_z is continuous on $L_H^2(W, \mu)$. The set of all a-weights on W will be denoted by $\mathcal{AW}(W)$.

The definition of admissible weight provides us with existence and uniqueness of the related Bergman kernel and completeness of the space $L_H^2(W, \mu)$. The concept of a-weight was introduced in [13], and in [14] several theorems concerning admissible weights are proved. An illustrative result is:

Theorem 2.2. [14, Cor. 3.1] *Let $\mu \in \mathcal{W}(W)$. If the function μ^{-a} is locally integrable on W for some $a > 0$ then $\mu \in \mathcal{AW}(W)$.*

Now, let us fix a point $t \in W$ and minimize the norm $\|f\|_\mu$ in the class $E_t = \{f \in L_H^2(W, \mu); f(t) = 1\}$. It can be proved, in a fashion similar to the classical case, that if μ is an admissible weight then there exists exactly one

function minimizing the norm. Let us denote it by $\phi_\mu(z, t)$. The *weighted Bergman kernel function* $K_{W, \mu}$ is defined as follows :

$$K_{W, \mu}(z, t) = \frac{\phi_\mu(z, t)}{\|\phi_\mu\|_\mu^2}.$$

3 From the Unweighted to the Weighted Case

It is well known that a Green's function for the Laplace operator takes the form

$$G_W(z, w) = h_W(z, w) - \ln |z - w|,$$

where h_W is harmonic w.r.t $z \in W$. Thus

$$\frac{\partial^2 G_W}{\partial z \partial \bar{w}} = \frac{\partial}{\partial z} \left(\frac{\partial h_W}{\partial \bar{w}} - \frac{1}{2} \frac{\partial}{\partial \bar{w}} [\ln(z - w) + \ln(\bar{z} - \bar{w})] \right) = \frac{\partial^2 h_W}{\partial z \partial \bar{w}}$$

Moreover $h_W(z, w) = G_W(z, w) + \ln |z - w| = G_W(w, z) + \ln |w - z| = h_W(w, z)$. Thus h_W is harmonic with respect to z and w . It turns out that (similarly as in the classical case (see [21])) regularity of ∂W is not important at all.

Theorem 3.1. *If $\rho(z) = |\mu(z)|^2$, where $\mu \in \text{Hol}(\overline{W})$, and has no zeros on \overline{W} , then*

$$K_{W, \rho}(z, w) = -\frac{2}{\pi \rho(z) \rho(w)} \frac{\partial^2}{\partial z \partial \bar{w}} G_{W, \rho}(z, w).$$

Proof. It is well known that any domain $W \subset \mathbb{C}$ may be written as

$$W = \bigcup_{j=1}^{\infty} W_j, \quad W_1 \Subset W_2 \Subset W_3 \Subset \dots,$$

where ∂W_j consists of a finite number of Jordan curves (we do not assume any regularity of ∂W), for any $j \in \mathbb{N}$. Let $\rho_j(z) = |\mu_j(z)|^2$ where $\mu_j \in \text{Hol}(\overline{W_j})$ are such that $\mu_j(z) \xrightarrow{j \rightarrow \infty} \mu(z)$ pointwise on W and $|\mu_j(z)|^2 \leq |\mu_{j+1}(z)|^2 \leq |\mu(z)|^2$ on W_j . We denote by $G_{W_j}(z, w)$ the Green's function of W_j and $K_{W_j, \rho_j}(z, w)$ the weighted Bergman kernel of W_j , $j \in \mathbb{N}$. Under these assumptions, $K_{W_j, \rho_j}(z, w) \rightarrow K_{W, \rho}$ locally uniformly on $W \times W$ (see [22]). The proof of the classical equality for the unweighted Bergman kernel and the Green's function of the Laplace operator given in [21] is based on two steps :

Step 1 : $(h_{W_j})_{j=1}^{\infty}$ is convergent to h_W in $W \times W$

Step 2 : $\left(\frac{\partial^2 h_{W_j}}{\partial z \partial \bar{w}} \right)_{j=1}^{\infty}$ is convergent to $\frac{\partial^2 h_W}{\partial z \partial \bar{w}}$ in $W \times W$.

The proof of Step 1 (given in [21]) is based on the fact, that a sequence $(h_{W_j})_{j=1}^\infty$ is increasing (since $h_{W_j}(z, w) = G_{W_j}(z, w) + \ln|z - w|$) and bounded from above by $h_W(z, w)$. Thus, by the Harnack theorem, it is convergent to a harmonic function \tilde{h} . It is shown that $h_W = \tilde{h}$, using the fact that 0 is the greatest harmonic minorant of G_{W_j} and h_{W_j} is the least harmonic majorant of $\ln|z - w|$ on W (see [7]). The proof of step 2 is based on the Poisson formula:

$$h_{W_j}(z, w) = \frac{1}{(2\pi)^2} \int \int_{[0, 2\pi]^2} h_{W_j}(z_0 + re^{it}, w_0 + re^{is}) \operatorname{Re} \left(\frac{re^{it} + z}{re^{it} - z} \right) \operatorname{Re} \left(\frac{re^{is} + w}{re^{is} - w} \right) dt ds$$

for $(z_0, w_0) \in W \times W$ and (z, w) in a neighborhood of (z_0, w_0) . Taking $\frac{\partial^2}{\partial z \partial \bar{w}}$ on both sides we get in the limit with $j \rightarrow \infty$ that

$$\lim_{j \rightarrow \infty} \frac{\partial^2 h_{W_j}(z, w)}{\partial z \partial \bar{w}} = \frac{\partial^2 h_W(z, w)}{\partial z \partial \bar{w}} = \frac{\partial^2 G_W(z, w)}{\partial z \partial \bar{w}}$$

We can simplify the proof of each of these assertions by using Harnack's theorem on harmonic functions. Taking $\lim_{j \rightarrow \infty}$ in

$$G_{W_j}(z, w) = h_{W_j}(z, w) - \ln|z - w|, \quad (3.1)$$

we get

$$h_W(z, w) - \ln|z - w| = G_W(z, w) = \lim_{j \rightarrow \infty} h_{W_j}(z, w) - \ln|z - w|,$$

thus $h_{W_j} \xrightarrow{j \rightarrow \infty} h_W$. Moreover, since $W_j \subset W_{j+1}$, we have

$$h_{W_j}(z, w) = G_{W_j}(z, w) + \ln|z - w| \leq G_{W_{j+1}}(z, w) + \ln|z - w| = h_{W_{j+1}}(z, w),$$

for $z, w \in W$. Similarly $h_{W_j}(z, w) \leq h_W(z, w)$, for $z, w \in W$. A nondecreasing sequence of harmonic functions

$$h_{W_1} \leq h_{W_2} \leq h_{W_3} \leq \dots \leq h_W$$

is convergent locally uniformly in W to a (harmonic function) h_W by Harnack's theorem, thus (again by Harnack's theorem) $h_{W_j} \rightarrow h_W$ in $C^\infty(W)$. So

$$\lim_{j \rightarrow \infty} \frac{\partial^2 h_{W_j}}{\partial z \partial \bar{w}} = \frac{\partial^2 h_W}{\partial z \partial \bar{w}}.$$

One can find in ([6], p.494) that $G_{W_j, \rho_j}(z, w) = \overline{\mu_j(z)} \mu_j(w) G_{W_j}(z, w)$. We conclude that

Lemma 3.2. *If $L_H^2(W_j) \neq \{0\}$, then G_{W_j, ρ_j} exists (and is given by (3.1)).*

Now let us observe that

$$\begin{aligned}
\frac{\partial^2 G_{W_j, \rho_j}(z, w)}{\partial z \partial \bar{w}} &= \frac{\partial^2}{\partial z \partial \bar{w}} \left(\overline{\mu_j(z)} \mu_j(w) G_{W_j}(z, w) \right) \\
&= \frac{\partial}{\partial z} \left(\overline{\mu_j(z)} \left(\frac{\partial \mu_j(w)}{\partial \bar{w}} G_{W_j}(z, w) + \mu_j(w) \frac{\partial G_{W_j}(z, w)}{\partial \bar{w}} \right) \right) \\
&= \frac{\partial}{\partial z} \left(\overline{\mu_j(z)} \mu_j(w) \frac{\partial G_{W_j}(z, w)}{\partial \bar{w}} \right) \\
&= \overline{\mu_j(z)} \mu_j(w) \frac{\partial^2 G_{W_j}(z, w)}{\partial z \partial \bar{w}}.
\end{aligned}$$

By the regularity of any ∂W_j we have

$$K_{W_j, \rho_j}(z, w) = -\frac{2}{\pi \rho_j(z) \rho_j(w)} \frac{\partial^2}{\partial z \partial \bar{w}} G_{W_j, \rho_j}(z, w),$$

which in the limit as $j \rightarrow \infty$ yields

$$\begin{aligned}
K_{W, \rho}(z, w) &= -\frac{2}{\pi \rho(z) \rho(w)} \overline{\mu(z)} \mu(w) \frac{\partial^2}{\partial z \partial \bar{w}} G_W(z, w) \\
&= -\frac{2}{\pi \rho(z) \rho(w)} \frac{\partial^2 G_{W, \rho}(z, w)}{\partial z \partial \bar{w}}.
\end{aligned}$$

□

3.1 Non-Holomorphic Weights

On closer scrutiny, the crucial thing in the proof of Theorem 3.1 was to relate the weighted Green function to the unweighted one. This relationship turns out to be preserved even if we relax the assumption about holomorphicity of weight, as the following reveals:

Theorem 3.3. *If $\rho(z) = |\mu(z)|^2$, where μ is a continuously differentiable function of x and y on a neighborhood of \overline{W} (and has no zeros on \overline{W}) then*

$$K_{W, \rho}(z, w) = -\frac{2}{\pi \rho(z) \rho(w)} \frac{\partial^2}{\partial z \partial \bar{w}} G_{W, \rho}(z, w).$$

Proof. Let $\{W_j\}_{j=1}^\infty$ and ρ_j be as in the proof of Theorem 3.1. The crucial thing is to find $g_j(z)$ such that $u_j(w) = g_j(w) U_j(w)$ is a general solution of the equation

$$\frac{\partial}{\partial \bar{w}} \frac{1}{\rho_j(w)} \frac{\partial}{\partial w} u_j(w) = 0,$$

and $U_j(w)$ is (an arbitrary) complex and harmonic on W_j (we define g on the same way by means of P_ρ). Thus

$$\begin{aligned}
0 &= \frac{\partial}{\partial \bar{w}} \frac{1}{\rho_j(w)} \frac{\partial}{\partial w} u_j(w) = \frac{\partial}{\partial \bar{w}} \frac{1}{\mu_j \bar{\mu}_j} \frac{\partial}{\partial w} (g_j(w) U_j(w)) \\
&= \frac{\partial}{\partial \bar{w}} \left(\frac{1}{\mu_j \bar{\mu}_j} \frac{\partial g_j}{\partial w} U_j \right) + \frac{\partial}{\partial \bar{w}} \left(\frac{1}{\mu_j \bar{\mu}_j} g_j \frac{\partial U_j}{\partial w} \right) \\
&= \left(\frac{\partial}{\partial \bar{w}} \frac{1}{\mu_j \bar{\mu}_j} \right) \frac{\partial g_j}{\partial w} U_j + \frac{1}{\mu_j \bar{\mu}_j} \left(\frac{\partial^2 g_j}{\partial \bar{w} \partial w} U_j + \frac{\partial g_j}{\partial w} \frac{\partial U_j}{\partial \bar{w}} \right) + \left(\frac{\partial}{\partial \bar{w}} \frac{1}{\mu_j \bar{\mu}_j} \right) g_j \frac{\partial U_j}{\partial w} \\
&+ \frac{1}{\mu_j \bar{\mu}_j} \left(\frac{\partial g_j}{\partial \bar{w}} \frac{\partial U_j}{\partial w} + g_j \underbrace{\frac{\partial^2 U_j}{\partial \bar{w} \partial w}}_0 \right)
\end{aligned}$$

Thus

$$\left\{ \begin{array}{l} \left(\frac{\partial}{\partial \bar{w}} \frac{1}{\mu_j \bar{\mu}_j} \right) \frac{\partial g_j}{\partial w} + \frac{1}{\mu_j \bar{\mu}_j} \frac{\partial^2 g_j}{\partial \bar{w} \partial w} = 0 \\ \frac{\partial g_j}{\partial w} = 0 \\ \left(\frac{\partial}{\partial \bar{w}} \frac{1}{\mu_j \bar{\mu}_j} \right) g_j + \frac{1}{\mu_j \bar{\mu}_j} \frac{\partial g_j}{\partial \bar{w}} = 0 \end{array} \right.$$

Remark 3.4. By the equation above, g_j is an antiholomorphic function.

Examining the system above, we see that the first equation is a consequence of the second one. Let us focus on the third one:

$$\left(\frac{\partial}{\partial \bar{w}} \frac{1}{\mu_j \bar{\mu}_j} \right) g_j + \frac{1}{\mu_j \bar{\mu}_j} \frac{\partial g_j}{\partial \bar{w}} = 0.$$

It may be written in the form

$$\frac{1}{g_j} \frac{\partial g_j}{\partial \bar{w}} = \frac{1}{\mu_j \bar{\mu}_j} \frac{\partial}{\partial \bar{w}} (\mu_j \bar{\mu}_j)$$

Thus, for a given μ_j , there is a function g_j which must satisfy :

$$\left\{ \begin{array}{l} \frac{1}{g_j} \frac{\partial g_j}{\partial \bar{w}} = \frac{1}{\mu_j \bar{\mu}_j} \frac{\partial}{\partial \bar{w}} (\mu_j \bar{\mu}_j) \\ \frac{\partial g_j}{\partial w} = 0 \end{array} \right.$$

Notice that, if μ_j is holomorphic and $g_j = \bar{\mu}_j$, then the system above is satisfied (in this case we get the result of [6]). We may proceed to get the exact form of $g_j(z)$, namely :

$$\begin{aligned}
\frac{\partial}{\partial \bar{w}} \ln g_j &= \frac{\partial}{\partial \bar{w}} \ln(\mu_j \bar{\mu}_j) \\
\ln g_j &= \ln(\mu_j \bar{\mu}_j) + h_j(w) \\
g_j(w) &= \mu_j \bar{\mu}_j e^{h_j(w)} = |\mu_j(w)|^2 e^{h_j(w)}
\end{aligned}$$

where $h_j \in C^1(\overline{W_j})$. But g_j is antiholomorphic, so

$$\begin{aligned} 0 &= \frac{\partial g_j}{\partial w} = \frac{\partial}{\partial w}(\mu_j \overline{\mu_j}) e_j^h + \mu_j \overline{\mu_j} e^h \frac{\partial h_j}{\partial w} \\ 0 &= \frac{\partial}{\partial w}(\mu_j \overline{\mu_j}) + \mu_j \overline{\mu_j} \frac{\partial h_j}{\partial w} \end{aligned}$$

So to sum up

$$g_j(z) = |\mu_j(z)|^2 e^{h_j(z)},$$

where

$$\frac{\partial}{\partial w}(\mu_j \overline{\mu_j}) + \mu_j \overline{\mu_j} \frac{\partial h_j}{\partial w} = 0.$$

In fact, we may choose h_j in such a way that $h_j \rightarrow h$, where $h \in C^1(\overline{W})$ (because such a g_j will still be a good solution to the posed problem). Then $g_j \rightarrow g = |\mu(z)|^2 e^{h(z)}$.

Again (as in [6])

$$G_{W_j, \rho_j}(z, w) = g_j(z) \overline{g_j(w)} G_{W_j}(z, w),$$

so as previously

$$\frac{\partial^2 G_{W_j, \rho_j}(z, w)}{\partial z \partial \overline{w}} = g_j(z) \overline{g_j(w)} \frac{\partial^2 G_{W_j}(z, w)}{\partial z \partial \overline{w}}$$

and

$$K_{W, \rho}(z, w) = -\frac{2}{\pi \rho(z) \rho(w)} \frac{\partial^2 G_{W, \rho}(z, w)}{\partial z \partial \overline{w}}.$$

□

4 Concluding Remarks

It is well established that weighted Bergman spaces are both intrinsically interesting and a powerful analytic tool. Our purpose in this paper has been to develop this set of ideas, and particularly the connection between the Bergman kernel and the Green's function in the weighted context. Some of the applications might be :

- a) With the established connection between weighted Bergman kernel and Green's function in hand, we can reformulate the weighted version of the so called "small conjecture" (it is the so called Skwarczyński distance equivalent to the Bergman distance, see [19, 18]) as :

Remark 4.1. Assume $W \Subset \mathbb{C}$, and μ is a continuously differentiable function of x and y on a neighborhood of \overline{W} . Then $t_n \rightarrow t \in \partial W$ represents defective evaluation iff

$$-\frac{2}{\pi \rho(z) \rho(w)} \frac{\partial^2}{\partial z \partial \overline{w}} G_{W, \rho}(z, w)(\cdot, t_n) \rightarrow \gamma$$

weakly in $L_H^2(W, \mu)$ and

$$-\frac{2}{\pi\rho(z)\rho(w)}\frac{\partial^2}{\partial z\partial\bar{w}}G_{W,\rho}(z,w)(t_n,t_n)\rightarrow\kappa^2$$

where $||\gamma|| \neq \kappa$. This is important, since the involved so called Skwarzynski distance is biholomorphically invariant, and given more explicitly than the Bergman distance.

- b) Using the method of alternating projections (see [17]), we can recover (having some Dirichlet and Neuman boundary conditions on G_W) $G_{W,\mu}$ for an arbitrary domain W lying in \mathbb{C} .

References

- [1] Bergman S. The kernel function and conformal mapping. A.M.S. Survey Number V, 2nd Edition, Providence, RI; 1970.
- [2] Bergman, S.; Schiffer, M. Kernel functions and conformal mapping. *Compositio Math.* 1951;8:205–249.
- [3] Carleson L. Selected problems on exceptional sets. Princeton : Van Nostrand; 1967.
- [4] Engliš M. Toeplitz operators and weighted Bergman kernels. *J. Funct. Anal.* 2008;255,no.6:1419–1457.
- [5] Engliš M. Weighted Bergman kernels and quantization. *Comm. Math. Phys.* 2002;227,no.2:211–241.
- [6] Garabedian P. R. A partial differential equation arising in conformal mapping. *Pacific J. Math.* 1951;1:485–524.
- [7] Helms L. Potential Theory. Universitext: Springer-Verlag London; 2009.
- [8] Krantz S. G. A new proof and a generalization of Ramadanov’s theorem. *Complex Variables and Elliptic Equations.* 2006;51,no.12:1125–1128.
- [9] Krantz S. G. Function Theory of Several Complex Variables. AMS Chelsea Publishing: Providence, RI; 2001.
- [10] Krantz S. G. Geometric analysis of the Bergman Kernel and Metric. Springer, New York; 2013.
- [11] Myrberg P. Über die existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche. *Acta Math.* 1933;61:39–79.

- [12] Odziejewicz A. On reproducing kernels and quantization of states. *Comm. Math. Phys.* 1988;114:577–597.
- [13] Pasternak-Winiarski Z. On the Wependence of the Reproducing Kernel on the Weight of Integration. *J. Funct. Anal.* 1990;94,no.1:110–134.
- [14] Pasternak-Winiarski Z. On weights which admit the reproducing kernel of Bergman type. *Internat. J. Math & Math. Sci.* 1992;15,no.1:1–14.
- [15] Ramadanov I. Sur une propriété de la fonction de Bergman (French). *C. R. Acad. Bulgare Sci.* 1967;20:759–762.
- [16] Shabat B.V. Introduction to Complex Analysis, Part II: Functions of Several Variables. Translations of Mathematical Monographs : Volume 110, American Mathematical Society, Providence, RI; 1992.
- [17] Skwarczyński M. Alternating projections in complex analysis. *Complex analysis and applications (Varna, 1983)*. Publ. House Bulgar. Acad. Sci., Sofia 1985;192–199.
- [18] Skwarczyński M. Biholomorphic invariants related to the Bergman function. Warsaw, Polish Scientific Publishing Company: *Wissertationes Mathematicae (Rozprawy Matematyczne)*. 1980;173.
- [19] Skwarczyński M. Metric completeness in holomorphic geometry. *Bull. Soc. Sci. Lett. Łódź Sér. Rech. Wéform.* 2006;51:11–22.
- [20] Skwarczyński M., Mazur T. Wstępne twierdzenia teorii funkcji wielu zmiennych zespolonych (Polish), Krzysztof Biesaga, Warszawa; 2001.
- [21] Suszczyński A. O pewnej tożsamości dla funkcji Greena i Bergmana (Polish). *Zeszyty Naukowe Filii Uniwersytetu Warszawskiego w Białymstoku*. 1984;39:113–118.
- [22] Wójcicki P.M. Weighted Bergman kernel function, admissible weights and the Ramadanov theorem. *Matematychni Studii*. 2014;42,no.2:160–164.

Steven G. Krantz
 Department of Mathematics
 Washington University in St. Louis
 St. Louis, Missouri, 63130 USA
 E-mail: sk@math.wustl.edu

Paweł M. Wójcicki
 Faculty of Mathematics and Information Science
 Warsaw University of Technology

Koszykowa 75, 00-662 Warsaw, Poland
E-mail: p.wojcicki@mini.pw.edu.pl